Almost and Mild Normality

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Abstract

Abstract: A topological space X is called *almost normal* if for any two disjoint closed subsets A and B of X one of which is regularly closed, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. We will present an example of a Tychonoff almost normal space which is not normal. Almost normality is not productive. We will present some conditions to assure that the product of two spaces will be almost normal.

Keywords: κ-normal; mildly normal; almost normal; regularly closed; normal.

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We investigate in this paper a weaker version of normality called *almost normality.* We will prove that almost normality is a property which lies between mild normality and normality. We will present an example of a Tychonoff almost normal space which is not normal. We will show that almost normality is not productive and we will present some conditions to assure that the product of two spaces will be almost normal.

We will denote an order pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space. And a Tychonoff space is a T_1 completely regular space. The interior of a set A will be denoted by int$A$, and the closure of a set A will be denoted by $\overline{A}$.

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1 Definition:

A subset A of a topological space X is called *regularly closed* (called also, *closed domain*) if $A = \overline{\operatorname{int} A}$. A subset A is called *regularly open* (called also, *open domain*) if $A = \operatorname{int}(\overline{A})$. Two subsets A and B in a topological space X are said to be separated if there exist two disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$.

2 Definition:

A topological space X is called *mildly normal* (called also *κ-normal)* if for any two disjoint regularly closed subsets A and B of X, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

3 Definition: (Singal and Arya)

A topological space X is called *almost normal* if for any two disjoint closed subsets A and B of X one of which is regularly closed, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

It is clear from the definitions that any normal space is almost normal and any almost normal space is mildly normal. The converse is not always true. The space $\omega_1 \times \omega_1 + 1$ is mildly normal, see [2] and [3], but not almost normal because the closed subset $A = \omega_1 \times \{\omega_1\}$ is disjoint from the regularly closed subset $B = \{\langle \alpha, \alpha \rangle : \alpha < \omega_1\}$ and they cannot be separated by two disjoint open subsets, see [1].

In [4], Singal and Arya introduced a finite space which is almost normal but not normal nor T_1. Since a T_1 finite space is discrete, the question, now, is the following: *Is there a Tychonoff space which is almost normal but not normal?* We will answer this below.

The following theorem, see [4], gives a characterization of almost normality which we will use.

4 Theorem: (Singal and Arya)

For a space X, the following are equivalent

1. X is almost normal.
2. For every closed set B and every regularly open set A containing B, there exists an open set U such that $B \subseteq U \subseteq \overline{U} \subseteq A$. ■

We will present an example of a Tychonoff space which is almost normal but not normal. But first we need to give a property which implies almost normality. Recall that a space X is *extremally disconnected* if it is T_1 and the closure of any open set is open. Many topologists required T_1 in the definition of extremally disconnected. So, we give the following weaker condition.

5 Definition:

A space X is called *weakly extremally disconnected* if the closure of any open set is open.

It is clear that any extremally disconnected space is weakly extremally disconnected. The converse is not always true. For example, let $\mathcal{T}_{\sqrt{2}} = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : \sqrt{2} \in U\}$, then $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is not T_1, as any open set containing 0 must contains $\sqrt{2}$, but the closure of any non-empty open set is $\mathbb{R}$, as $\{\sqrt{2}\}$ is dense in $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$. Thus $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is weakly extremally disconnected but not extremally disconnected.

The next theorem is clear because in weakly extremally disconnected spaces any regular closed set is clopen. Note that we do not assume any separation axiom.

6 Theorem:

Any weakly extremally disconnected space is almost normal.

Now, we give an example of a Tychonoff space which is almost normal but not normal.

7 Example:

Arrange all rationals of the closed unit interval $I = [0, 1]$ into a sequence $\{q_1, q_2, q_3, ...\}$. Let $t \in I$, then there is a subsequence $\{q_{n_1}, q_{n_2}, q_{n_3}, ...\}$ that converges to t in the usual topology. Let $N_t = \{n_1, n_2, n_3, ...\}$. It is clear that $N_t \neq N_{t'}$ for all $t, t' \in I$ with $t \neq t'$. The family $\{U_t : t \in I\}$, where $U_t = (\beta\mathbb{N} \setminus \mathbb{N}) \cap \overline{N_t}$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$ with the discrete topology, has cardinality continuum c and consists of non-empty subsets of $\beta\mathbb{N} \setminus \mathbb{N}$. For every $t \neq t' \in I$, we have $N_{t'} = F \cup M$, where $F \subset \mathbb{N}$ is a finite set and $M \cap N_t = \emptyset$. Thus, see [1,3.6.4], we have $\overline{M} \cap \overline{N_t} = \emptyset$. Since $F = \overline{F} \subset \mathbb{N}$, then $U_t \cap U_{t'} = (\beta\mathbb{N} \setminus \mathbb{N}) \cap \overline{N_t} \cap \overline{N_{t'}} = (\beta\mathbb{N} \setminus \mathbb{N}) \cap N_t \cap (\overline{F} \cup \overline{M}) = (\beta\mathbb{N} \setminus \mathbb{N}) \cap ((\overline{N_t} \cap \overline{F}) \cup (\overline{N_t} \cap \overline{M})) \subseteq (\beta\mathbb{N} \setminus \mathbb{N}) \cap \mathbb{N} = \emptyset$.

Now, for each $t \in I$, choose a point $x_t \in U_t$ and define $X = \mathbb{N} \cup \{x_t : t \in I\}$. Since $\mathbb{N}$ is locally compact and dense in X, then, see [1, 3.3.9], $\mathbb{N}$ is open in X, hence $X \setminus \mathbb{N}$ is closed in X consisting of isolated points. Thus, by Jones's Lemma, X is not normal. Since X is a subspace of $\beta\mathbb{N}$ and $\beta\mathbb{N}$ is Hausdorff and compact, then X is Tychonoff. Since $\beta\mathbb{N}$ is extremally disconnected and X is dense in $\beta\mathbb{N}$, as $\mathbb{N} \subset X$ and $\mathbb{N}$ is dense in any compactification of it, then X is extremally disconnected (extremal disconnectedness is hereditary with respect to both open subsets and dense subsets, see [1]). Therefore, X is almost normal Tychonoff space which is not normal. ■

Any T_4 space which is not extremally disconnected is an example of an almost normal space which is not weakly extremally disconnected. It is natural to ask the following problem. "*Is there a Tychonoff space which is almost normal but not weakly extremally disconnected nor normal?*" The answer is yes. Take the space X of Example 7 and consider the free sum $X \oplus \mathbb{R}$, where $\mathbb{R}$ is considered with its usual metric topology.

Now, we will give an example of a Hausdorff almost normal space X which is not regular nor normal. First, let us recall some basics of the notion of filters. A *filter* on a set X is a collection F of subsets of X with the following properties:

1. Every subset of X which contains a set of F belongs to F.
2. Every finite intersection of sets of F belongs to F.
3. The empty set is not in F.

If a filter F on X has the property that there is no filter F' on X such that $F \subseteq F'$ and $F \neq F'$, then F is called an *ultrafilter* on X. Equivalently, F is an ultrafilter if and only if for every two disjoint subsets A and B of X such that $A \cup B \in F$, then either $A \in F$ or $B \in F$. If a point x is in all sets of a filter we call it a *cluster* point. Clearly an ultrafilter can have at most one cluster point. An ultrafilter with a cluster point p is just the set of all sets containing p and is called a *principal* ultrafilter. An ultrafilter with no cluster point is called *nonprincipal* or *free* ultrafilter.

For more details about the next example, see [5].

8 Example:

Let $\mathcal{M}$ be the collection of all free ultrafilters on $\mathbb{N}$. Let $X = \mathbb{N} \cup \mathcal{M}$. Let the topology $\mathcal{T}$ on X be generated by the neighborhood system $\{\mathcal{B}(x) : x \in X\}$ where $\mathcal{B}(x) = \{\{x\}\}$ for each $x \in \mathbb{N}$, i.e., points of $\mathbb{N}$ are isolated, and $\mathcal{B}(x) = \{A \cup \{F\} : A \in F \in \mathcal{M}\}$ for each $x = F \in \mathcal{M}$.

X is Hausdorff because any two members F and G of $\mathcal{M}$, being ultrafilters, are incomparable. So, there exist $A \in F \setminus G$, $B \in G \setminus F$. Then since F is an ultrafilter, $\mathbb{N} \setminus B \in F$, so $A \cap (\mathbb{N} \setminus B) = A \setminus B$. Similarly, $B \setminus A \in G$. Thus $(A \setminus B) \cup \{F\}$ and $(B \setminus A) \cup \{G\}$ are disjoint open neighborhood of F and G. Note that $F \in \mathcal{M}$ can be separated from any $y \in \mathbb{N}$ precisely since no y can be contained in all sets of F because F can have no cluster points.

For extremal disconnectedness, suppose p is a limit point of an open set U which does not belong to U. Since each point of $\mathbb{N}$ is open, $p \in X \setminus \mathbb{N} = \mathcal{M}$. So p is an ultrafilter, say F, and every neighborhood $A \cup \{F\}$ of $p = F$ (where $A \in F$) intersects U. But since F itself does not belong to U, this intersection is contained in $\mathbb{N}$. Thus, $U \cap \mathbb{N}$ intersects every member of the ultrafilter F, but it is a property of ultrafilters that for every subset $S \subset \mathbb{N}$, either S or its complement belongs to the ultrafilter. Since $U \cap \mathbb{N}$ does not intersect its own complement, $U \cap \mathbb{N}$ itself must belong to the ultrafilter F. That is, $(U \cap \mathbb{N}) \in p$. Thus $(U \cap \mathbb{N}) \cup \{F\}$, or equivelantly, $(U \cap \mathbb{N}) \cup \{p\}$, is open. Thus $U \cup \{p\} = U \cup ((U \cap \mathbb{N}) \cup \{p\})$ is open, and since p was an arbitrary limit point of U, $\overline{U}$ must be open. Thus X is extremally disconnected.

Now, any basic open set of the form $A \cup \{F\}$ has a limit point every ultrafilter G which contains A as an element, for if $B \in G$ and $A \in G$, then $A \cap B \neq \emptyset$, so $B \cup \{G\} \cap A \cup \{F\} \neq \emptyset$. So, if $B \subset A$, the set $\overline{B \cup \{F\}}$ contains all ultrafilters which contain B, which means that $\overline{B \cup \{F\}}$ is not contained in $A \cup \{F\}$. Thus X cannot be regular. ■

9 Theorem:

If X is almost normal countably compact and M is paracompact first countable, then $X \times M$ is almost normal.

Proof: Let A and B be any two disjoint non-empty closed subsets of $X \times M$ where B is regularly closed. Let $p_1 : X \times M \longrightarrow X$ be the natural projection. For each $m \in M$ and each M-open neighborhood $U(m)$ of m, define the following subsets of X:

$$A_{U(m)} = \{x \in X : \text{ there exists } y \in U(m) \text{ such that } \langle x, y \rangle \in A\}$$

$$= p_1((X \times U(m)) \cap A).$$

$$B_{U(m)} = \{x \in X : \text{ there exists } z \in U(m) \text{ such that } \langle x, z \rangle \in \mathrm{int}B\}$$

$$= p_1((X \times U(m)) \cap \mathrm{int}B).$$

For each $m \in M$, fix a countable decreasing local base $\{U_n(m) : n \in \omega\}$ for M at m. We will write A_{U_n} instead of $A_{U_n}(m)$ and B_{U_n} instead of $B_{U_n}(m)$. For each $n \in \omega$ and each $m \in M$, we have $\overline{A_{U_n}} \cap \overline{B_{U_n}} \supseteq \overline{A_{U_{n+1}}} \cap \overline{B_{U_{n+1}}}$. Thus the family $\{\overline{A_{U_n}} \cap \overline{B_{U_n}} : n \in \omega\}$ is a decreasing sequence of closed subsets of X. If $\overline{A_{U_n}} \cap \overline{B_{U_n}} \neq \emptyset$ for each $n \in \omega$, then by countable compactness of X, there exists an $x \in X$ such that $x \in \bigcap_{n \in \omega} (\overline{A_{U_n}} \cap \overline{B_{U_n}})$. So, if W is any X-open neighborhood of x, then $W \cap A_{U_n} \neq \emptyset \neq W \cap B_{U_n}$ for each $n \in \omega$. We will show that $\langle x, m \rangle \in A \cap B$. Let $W \times U$ be any basic open neighborhood of $\langle x, m \rangle$ in $X \times M$. Then there exists an $n \in \omega$ such that $\langle x, m \rangle \in W \times U_n \subseteq W \times U$. Now, $W \cap B_{U_n} \neq \emptyset$ implies that $W \cap \{x \in X : \text{ there exists } z \in U(m) \text{ such that } \langle x, z \rangle \in \mathrm{int}B\} \neq \emptyset$, thus there exists $a \in W$ and $a \in \{x \in X : \text{ there exists } z \in U(m) \text{ such that } \langle x, z \rangle \in \mathrm{int}B\}$. That is, $a \in W$ and there exists $z \in U_n$ such that $\langle a, z \rangle \in \mathrm{int}B$, thus $\langle a, z \rangle \in (W \times U_n) \cap \mathrm{int}B$. Thus $(W \times U_n) \cap \mathrm{int}B \neq \emptyset$. Therefore, $\langle x, m \rangle \in$

$\overline{\operatorname{int} B} = B$. Also, $W \cap A_{U_n} \neq \emptyset$ implies $W \cap \{x \in X : \text{there exists } y \in U(m) \text{ such that } \langle x, y\rangle \in A\} \neq \emptyset$, thus there exists $a \in W$ and $a \in \{x \in X : \text{there exists } y \in U(m) \text{ such that } \langle x, y\rangle \in A\}$, which means $a \in W$ and there exists $y \in U_n$ such that $\langle a, y\rangle \in A$. Thus $\langle a, y\rangle \in (W \times U_n) \cap A$, i.e., $(W \times U_n) \cap A \neq \emptyset$. Thus $\langle x, m\rangle \in \overline{A} = A$. Thus $A \cap B \neq \emptyset$ which is a contradiction. Therefore, we conclude that for each $m \in M$ there exists an open neighborhood $U(m)$ of m such that $\overline{A_{U(m)}} \cap \overline{B_{U(m)}} = \emptyset$. Since the natural projection is an open function, then for each $m \in M$ we have that $\overline{B_{U(m)}}$ is a regularly closed subset of X. Since X is almost normal, then for each $m \in M$, there are open disjoint subsets G_m and H_m of X such that $\overline{A_{U(m)}} \subseteq G_m$ and $\overline{B_{U(m)}} \subseteq H_m$. Now, $\{U(m) : m \in M\}$ is an open cover of M. Since M is paracompact, then there is a locally finite open cover $\{V_m : m \in M\}$ such that for each $m \in M$, we have $V_m \subseteq \overline{V_m} \subseteq U_m = U(m)$.

Claim 1: $B \subseteq \bigcup_{m \in M}(H_m \times V_m)$.

Let $\langle x, y\rangle \in B$ be arbitrary, then there is an $m' \in M$ such that $y \in V_{m'} \subseteq \overline{V_{m'}} \subseteq U_{m'}$. Suppose that $x \notin \overline{B_{U'_m}}$, then there exists an open neighborhood G of x such that $G \cap B_{U_{m'}} = \emptyset$. By the definition of $B_{U_{m'}}$, we have $G \cap U_{m'} = \emptyset$. It means that for each $z \in U_{m'}$ and each $x' \in G$, we have $\langle x', z\rangle \notin \operatorname{int} B$. Therefore, $(G \times U_{m'}) \cap B = \emptyset$. Since $x \in G$ and $y \in V_{m'} \subseteq U_{m'}$, then $\langle x, y\rangle \notin \overline{\operatorname{int} B} = B$, a contradiction. Therefore, $x \in \overline{B_{U_{m'}}} \subseteq H_{m'}$, hence $\langle x, y\rangle \in H_{m'} \times V_{m'} \subseteq \bigcup_{m \in M}(H_m \times V_m)$ and hence Claim 1 is proved.

Claim 2: $A \cap \overline{(H_m \times V_m)} = \emptyset$ for each $m \in M$.

Suppose that there exists an $m \in M$ and $\langle x, y\rangle \in X \times M$ such that $\langle x, y\rangle \in A \cap \overline{(H_m \times V_m)} = A \cap (\overline{H_m} \times \overline{V_m})$. Then $y \in \overline{V_m}$. Since $\overline{V_m} \subseteq U_m$, then $A \cap (X \times \overline{V_m}) \subseteq A \cap (X \times U_m) \subseteq \overline{A \cap (X \times U_m)}$. So, by continuity of p_1, we have $p_1(A \cap (X \times \overline{V_m})) \subseteq p_1(\overline{A \cap (X \times U_m)}) \subseteq \overline{p_1(A \cap (X \times U_m))} = \overline{A_{U_m}}$. Since $y \in \overline{V_m}$ and $\langle x, y\rangle \in A$, then $x \in p_1(A \cap (X \times \overline{V_m})) \subseteq \overline{A_{U_m}} \subseteq G_m$. But x is also in $\overline{H_m}$, thus $G_m \cap \overline{H_m} \neq \emptyset$ which is a contradiction, and hence Claim 2 is proved.

Now, since $\{V_m : m \in M\}$ is locally finite, then $\{H_m \times V_m : m \in M\}$ is a locally finite family of open subsets of $X \times M$. By Claim 1, $B \subseteq \bigcup_{m \in M}(H_m \times V_m)$ where the later set is open. By Claim 2, $A \bigcap \overline{\bigcup_{m \in M}(H_m \times V_m)} = \emptyset$, because $\overline{\bigcup_{m \in M}(H_m \times V_m)} = \bigcup_{m \in M} \overline{(H_m \times V_m)}$ by locally finiteness.

Therefore A and B can be separated by disjoint open sets, thus $X \times M$ is almost normal. ■

The space $\omega_1 \times \omega_1 + 1$ shows that neither paracompactness nor first countable can be dropped from the hypotheses on the second factor.

10 Corollary:

If X is almost normal countably compact and M is metrizable, then $X \times M$ is almost normal. ■

We still do not know if the Sorgenfrey line square is almost normal nor if the Niemytzki (the Moore) plane is almost normal.

Now, let $\mathbb{Q}$ denote the set of rational numbers and $\mathbb{P}$ denote the set of irrational numbers. Let $\mathbf{M}$ denote the Michael line. So, $\mathbf{M} = \mathbb{R}$, the irrational points are isolated, and a basic open neighborhood for a rational point is the same as in $\mathbb{R}$ with the usual topology. It is well known that $\mathbf{M} \times \mathbb{P}$ is not normal, where the topology on $\mathbb{P}$ is the usual topology, see [En, 5.1.32]. We are going to show that $\mathbf{M} \times \mathbb{P}$ is not almost normal.

11 Proposition:

The product space $\mathbf{M} \times \mathbb{P}$ is not almost normal.

Proof:

Let $U = \{\langle x, y\rangle : x \in \mathbb{R}, y \in \mathbb{P}, x > y\}$ and $V = \{\langle x, y\rangle : x \in \mathbb{R}, y \in \mathbb{P}, x < y\}$. Then U and V are two disjoint open sets in $\mathbf{M} \times \mathbb{P}$, and

$$\mathbf{M} \times \mathbb{P} = U \cup V \cup \{\langle p, p\rangle : p \in \mathbb{P}\}.$$

Let $\mathbf{L} = \{\langle x, x\rangle : x \in \mathbb{R}\}$. Then,

$$\mathbf{L} \cap (\mathbf{M} \times \mathbb{P}) = \{\langle p, p\rangle : p \in \mathbb{P}\}$$

.Hence

$$\overline{U} \setminus U = \{\langle p, p\rangle : p \in \mathbb{P}\}.$$

Since the closure of any open set is regularly closed, then $A = \overline{U} = U \cup \{\langle p, p\rangle : p \in \mathbb{P}\}$ is a regularly closed set. Now, let $B = (\mathbb{Q} \times \mathbb{P}) \cap V$. We want to show that B is a closed subset in $\mathbf{M} \times \mathbb{P}$ or equivalently, $(\mathbf{M} \times \mathbb{P}) \setminus B$ is an open set. Let $\langle x, y\rangle \in \mathbf{M} \times \mathbb{P}$ and $\langle x, y\rangle \notin B$ we have the following cases:

case(1) $\langle x, y\rangle \in U$. Since $U \cap V = \emptyset$, then $U \cap B = \emptyset$. Since U is an open set contains $\langle x, y\rangle$, then there exists open neighborhood $W \subseteq U$ of $\langle x, y\rangle$ such that $W \subseteq (\mathbf{M} \times \mathbb{P}) \setminus B$. Hence $\langle x, y\rangle \in int((\mathbf{M} \times \mathbb{P}) \setminus B)$.

case(2) $\langle x, y\rangle \in \mathbf{L}$. Then $x, y \in \mathbb{P}$ and $x = y$. Since the subset $\mathbb{P} \times \mathbb{P} = \{\langle x, y\rangle :, x, y \in \mathbb{P}\}$ is open set, there exist open neighborhood W of $\langle x, y\rangle$, such that $W \subseteq \mathbb{P} \times \mathbb{P}$. So, $W \cap (\mathbb{Q} \times \mathbb{P}) = \emptyset$. Since $B \subseteq (\mathbb{Q} \times \mathbb{P})$, then $W \subseteq (\mathbf{M} \times \mathbb{P}) \setminus B$. Hence $\langle x, y\rangle \in int((\mathbf{M} \times \mathbb{P}) \setminus B)$.

case(3) $\langle x, y\rangle \in V \setminus B$. Since $V \setminus B \subseteq \mathbb{P} \times \mathbb{P}$, then similarly case(2) there exists open neighborhood W of $\langle x, y\rangle$ such that $W \subseteq \mathbb{P} \times \mathbb{P}$. So, $W \subseteq (\mathbf{M} \times \mathbb{P}) \setminus B$. Hence $\langle x, y\rangle \in int((\mathbf{M} \times \mathbb{P}) \setminus B)$.

From the above we have $(\mathbf{M}\times\mathbb{P})\setminus B$ is open set. Hence B is a closed set. Since $\overline{U}\cap V=\emptyset$, and $B\subseteq V$, then $\overline{U}\cap B=\emptyset$ i.e. $A\cap B=\emptyset$, where A is a regularly closed set and B is a closed set.

Now, for any point $\langle x,y\rangle\in\mathbf{M}\times\mathbb{P}$ we shall let $D(\langle x,y\rangle,r)$ denote the basic open neighborhood of $\langle x,y\rangle$ with center $\langle x,y\rangle$ and radius r define as follows: when $x\in\mathbb{Q}$ let

$$D(\langle x,y\rangle,r)=((x-r,x+r)\times(y-r,y+r))\cap(\mathbf{M}\times\mathbb{P}).$$

When $x\in\mathbb{P}$, let

$$D(\langle x,y\rangle,r)=(\{x\}\times(y-r,y+r))\cap(\mathbf{M}\times\mathbb{P})$$

Now, Let W_1 and W_2 be any two open sets such that $A\subseteq W_1$ and $B\subseteq W_2$. We will prove that $W_1\cap W_2\neq\emptyset$. Now, for each $\langle p,p\rangle\in A\subseteq W_1$, there exists a basic open set $D(\langle p,p\rangle,r_p)$, where $r_p>0$ such that $\langle p,p\rangle\in D(\langle p,p\rangle,r_p)\subseteq W_1$. Now, define $S_n=\{\langle p,p\rangle : p\in\mathbb{P},\ r_p>1\setminus n\}$. It is clear that $S_n\subseteq A$ for all $n\in\mathbb{N}$. We need to show that

$$\bigcup_{n\in\mathbb{N}} S_n=\bigcup\{\langle p,p\rangle : p\in\mathbb{P}\}$$

Let $\langle p,p\rangle\in A$, where $p\in\mathbb{P}$. Then there is basic open set $D(\langle p,p\rangle,r_p)$ such that $\langle p,p\rangle\in D(\langle p,p\rangle,r_p)\subseteq W_1$. And so, there is $m\in\mathbb{N}$ such that $1/m<r_p$. Hence $\langle p,p\rangle\in S_m$. Thus

$$\bigcup_{n\in\mathbb{N}} S_n\subseteq\bigcup\{\langle p,p\rangle : p\in\mathbb{P}\}$$

It clear that

$$\bigcup_{n\in\mathbb{N}} S_n\supseteq\bigcup\{\langle p,p\rangle : p\in\mathbb{P}\}$$

Then

$$\bigcup_{n\in\mathbb{N}} S_n=\bigcup\{\langle p,p\rangle : p\in\mathbb{P}\}.$$

Now, we have $\mathbf{L}=\{\langle x,y\rangle : x,y\in\mathbb{R},x=y\}=(\bigcup_{n\in\mathbb{N}}S_n)\bigcup(\{\langle q,q\rangle : q\in\mathbb{Q}\})$ is a closed subset in $(\mathbb{R}^2,\mathcal{U})$. Hence $\mathbf{L}$ is a complete metric space as subspace in $(\mathbb{R}^2,\mathcal{U})$, where $\mathcal{U}$ is the usual metric topology on the plane $\mathbb{R}^2$. Since $\{\langle q,q\rangle : q\in\mathbb{Q}\}$ is countable set and $\{\langle q,q\rangle\}$ is nowhere dense for each $q\in\mathbb{Q}$ in $\mathbf{L}$ as subspace in $(\mathbb{R}^2,\mathcal{U})$. By Baire Category Theorem, there exists $n_0\in\mathbb{N}$ such that S_{n_0} is no nowhere dense, i.e. there exists basic open set $I\subseteq\mathbf{L}$ such that $S_{n_0}=\{\ \langle p,p\rangle :\ p\in\mathbb{P},\ r_p>1\setminus n_0\}$ is dense in I as subspace in $\mathbf{L}$ i.e. $I \subseteq \overline{S_{n_0}}^L$. It is well known that I is on the form $I = ((a,b) \times (a,b)) \cap \mathbf{L}$, where $a, b \in \mathbb{R}$. Now, let $q \in \mathbb{Q}$ such that $\langle q, q\rangle \in I$ and let $m_0 \in \mathbb{N}$ such that $0 < 1 \setminus m_0 < 1 \setminus 2n_0$. We will show that every open neighborhood for every $\langle q, p\rangle \in B$, where $p \in (\,q, q+1 \setminus m_0)$ must intersect W_1. So, W_1 and W_2 can not be disjoint. Let G be any basic open neighborhood of $\langle q, P\rangle \in B = (\mathbb{Q} \times \mathbb{P}) \cap V$ in $\mathbf{M} \times \mathbb{P}$, where $\langle q, q\rangle \in I$ and $p \in (\,q, q+1 \setminus m_0)$. We need to show that $G \cap W_1 \neq \emptyset$. Without loss generality we can assume that $G = D(\langle q, p\rangle, \epsilon)$, where $\epsilon > 0$. Since $I \subseteq \overline{S_{n_0}}^L$ i.e. $\overline{S_{n_0}}^L$ is dense in I as subspace in $\mathbf{L}$, therefore there exist $p_1 \in \mathbb{P}$ such that $\langle p_1, p_1\rangle \in (\,(\,q - \epsilon \setminus 2,\ q + \epsilon \setminus 2) \times (q - \epsilon \setminus 2, q + \epsilon \setminus 2)) \cap I$, and $\langle p_1, p_1\rangle \in S_{n_0}$, since $1 \setminus m < 1 \setminus 2n_0$ and $r_{p_1} > 1 \setminus n_0$. Hence $D(\langle p_1, p_1\rangle, r_{p_1}) \cap G \neq \emptyset$. Since $D(\langle p_1, p_1\rangle, r_{p_1}) \subseteq W_1$, then $W_1 \cap G \neq \emptyset$. Therefore $W_1 \cap W_2 \neq \emptyset$. Hence we cannot separated A and B by tow disjoint open sets. Thus $\mathbf{M} \times \mathbb{P}$ is not almost normal space.■

It is still unknown if the Michael product $\mathbf{M} \times \mathbb{P}$ is mildly normal or not, [3]. Also, whether the Dowker theorem version for almost normality is true or not, which is the following problem:" *If X is almost normal countably paracompact and Y is compact second countable, is then $X \times Y$ almost normal?*

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