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Elzaki and Sumudu Transforms for Solving Some Differential Equations

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Abstract

In this paper we propose a novel computational algorithm for solving ordinary differential equations with non-constants coefficients by using the modified version of Laplace and Sumudu transforms which is called Elzaki transform. Elzaki transform can be easily applied to the initial value problems with less computational work. The several illustrative examples can not solve by Sumudu transform, this means that Elzaki transform is a powerful tool for solving some ordinary differential equations with variable coefficients.

Keywords: Elzaki transform, Sumudu transform, Laplace transform, Differential equations.

Introduction

Elzaki transform [1,2,3,4], which is a modified general Laplace and Sumudu transforms, [1] has been shown to solve effectively, easily and accurately a large class of linear differential equations. Elzaki transform was successfully applied to integral equations, partial differential equations [2], ordinary differential equations with variable coefficients [4] and system of all these equations.

The purpose of this paper is to solve differential equations with variable coefficients which were not solved by Sumudu transform; this means that Sumudu transform failed to solve these types of differential equations.

Recently Tarig M. Elzaki [1, 2, 3, 4], introduced a new integral transform and named it as Elzaki transform that is defined by the integral equation:

$$E\left[f(t)\right]=u^2\int_0^\infty f(ut)e^{-t}dt=T(u), \qquad u\in(k_1,k_2)$$

Recall the following theorems that were given by Tarig Elzaki [1,2,3,4] where they discussed Elzaki transform of the derivatives.

Theorem (1)

Let $T(u)$ is ELzaki transform of $f(t)$ $\left[E(f(t))=T(u)\right]$. then:

$$(i)\quad \mathrm{E}\left[f'(t)\right]=\frac{T(u)}{u}-u\,f(0) \quad (ii)\quad \mathrm{E}\left[f''(t)\right]=\frac{T(u)}{u^2}-f(0)-uf'(0)$$

$$(iii)\quad \mathrm{E}\left[f^{(n)}(t)\right]=\frac{T(u)}{u^n}-\sum_{k=0}^{n-1}u^{2-n+k}f^{(k)}(0)$$

Proof

(i) $\mathrm{E}\left[f'(t)\right]=u\int_0^\infty f'(t)\,e^{\frac{-t}{u}}\,dt$ Integrating by parts to find that:

$$\mathrm{E}\left[f'(t)\right]=\frac{T(u)}{u}-u\,f(0)$$

(ii) Let $g(t)=f'(t)$, then: $\mathrm{E}\left[g'(t)\right]=\frac{1}{u}\ \mathrm{E}\left[g(t)\right]-ug(0)$

We find that, by using (i):

$$\mathrm{E}\left[f''(t)\right]=\frac{T(u)}{u^2}-f(0)-uf'(0)$$

(iii) Can be proof by mathematical induction.

Where that Sumudu transform of derivatives is given by:

$S[f'(t)]=\frac{1}{u}\left[F(u)-f(0)\right]$, and $S[f''(t)]=\frac{1}{u^2}\left[F(u)-f(0)-uf'(0)\right]$

Theorem (2)

Let $f(t)\in A=\left\{f(t)\,|\,\exists M,k_1,k_2>0, such\ \ that\left|f(t)\right|<Me^{|t|/k_i}\ ,\ if\ t\in(-1)^j\times[0,\infty)\right\}$

With Laplace transform $F(s)$, Then ELzaki transform $T(u)$ of $f(t)$ is given by:

$$T(u)=u\,F\left(\frac{1}{u}\right)$$

Proof

Let: $f(t) \in A$, then for $k_1 < u < k_2$ $T(u) = u^2 \int_0^\infty e^{-t} f(ut)\,dt$

Let $w = ut$ then we have:

$$T(u) = u^2 \int_0^\infty e^{-\frac{w}{u}} f(w) \frac{dw}{u} = u \int_0^\infty e^{-\frac{w}{u}} f(w)\,dw = u\,F\left(\frac{1}{u}\right).$$

Also we have that $T(1) = F(1)$ so that both the ELzaki and Laplace transforms must coincide at $u = s = 1$.

ELzaki Transform Multiple Shift Theorem

The discrete ELzaki transform can be used effectively to discern some rules on how the general transform affects various functional operations, Tarig M. ELzaki prove that

$$\mathrm{E}\left[tf'(t)\right] = u^2 \frac{d}{du}\left[\frac{T(u)}{u} - uf(0)\right] - u\left[\frac{T(u)}{u} - uf(0)\right].$$

And that:

$$\mathrm{E}\left[te^t\right] = \frac{u^3}{(1-u)^2}$$

One may ask how ELzaki transform acts on $t^n f(t)$. clearly, if $f(t) = \sum_{n=0}^{\infty} a_n t^n$, then

$$\mathrm{E}\left[tf(t)\right] = \sum_{n=0}^{\infty} (n+1)!\,a_n u^{n+3} = u^3 \frac{d}{du} \sum_{n=0}^{\infty} n!\,a_n u^{n+1}$$

Theorem 3

Let $T(u)$ be ELzaki transform of the function $f(t)$ in A, then ELzaki transform of the function $tf(t)$ is given by: $\mathrm{E}\left[tf(t)\right] = u^2 \frac{d}{du} T(u) - uT(u)$

Proof

The function $t\,f(t)$ is in A, since $f(t)$ is so: and integrating by parts we find that:

$$\frac{d}{du}T(u)= T'(u) \;=\frac{d}{du}\int_0^{\infty} u\,e^{-\frac{t}{u}} f(t)\,dt = \int_0^{\infty}\frac{\partial}{\partial u}\left[ue^{-\frac{t}{u}} f(t)\right]dt$$

$$= \int_0^{\infty}\frac{1}{u}e^{-\frac{t}{u}}\left(t\,f(t)\right)dt+\int_0^{\infty} e^{-\frac{t}{u}} f(t)\;dt = \frac{1}{u^2}\mathrm{E}\left[t\;f(t)\right]+\frac{1}{u}\,\mathrm{E}\left[f(t)\right]$$

Then we have: $\mathrm{E}\left[t\;f\;(t)\right]=\;u^2\frac{d}{du}T\;(u)-uT\;(u)$

In the general cases we can external theorem 3 as,

$$(i)\,\mathrm{E}\left[t\,f\,'(t)\right]=u^2\frac{d}{du}\left[\frac{T\;(u)}{u}-uf\;(0)\right]-u\left[\frac{T\;(u)}{u}-uf\;(0)\right]$$

$$(ii)\;\mathrm{E}\left[t^2 f\,'(t)\right]=u^4\frac{d^2}{du^2}\left[\frac{T\;(u)}{u}-uf\;(0)\right]$$

$$(iii)\;\mathrm{E}\left[t\,f\,''(t)\right]=u^2\frac{d}{du}\left[\frac{T\;(u)}{u^2}-f\;(0)-uf\,'(0)\right]-u\left[\frac{T\;(u)}{u}-f\;(0)-uf\,'(0)\right]$$

$$(iv)\;\mathrm{E}\left[t^2 f\,''(t)\right]=\;u^4\frac{d^2}{du^2}\left[\frac{T\;(u)}{u^2}-f\;(0)-uf\,'(0)\right]$$

The proof of these equations is easy, by using theorem 3.

And Sumudu transform of these is given by:

$$(i)\,S\left[tf(t)\right]=u^2\frac{d}{du}F(u)+uF(u)$$

$$(ii)\,S\left[t^2 f(t)\right]=u^4\frac{d^2}{du^2}F(u)+4u^3\frac{d}{du}F(u)+2u^2F(u)$$

Where $F(u)$ is the Sumudu transform of $f\;(t)$.

Example 1

Consider the second order initial value problem:

$$t^2y''+4ty'+2y\;=12t^2\;,\quad y(0)=y'(0)=0 \tag{1}$$

First we apply Sumudu transform to this equation to find:

$$u^2F''(u)+4uF'(u)+2F(u)=24u^2$$

Which is the same equation (1), this means that Sumudu transform can not solve this equation.

Now if apply Elzaki transform to equation (1) and make use of the initial conditions and above theorems, then we find:

$$u^4 \frac{d^2}{du^2}\left[\frac{T(u)}{u^2}\right] + 4u^2 \frac{d}{du}\left[\frac{T(u)}{u}\right] - 4u\left[\frac{T(u)}{u}\right] + 2T(u) = 24u^4 \quad or \quad T''(u) = 24u^2$$

The solution of the last equation is $T(u) = 2u^4 + c_1 u + c_2$, using the conditions to find $c_1 = c_2 = 0$, then $T(u) = 2u^4$.

By using the inverse Elzaki transform we find the solution in the form: $y(t) = t^2$.

Example 2

Consider the third order non-constant coefficients differential equation:

$$t^2 y''' + 6ty'' + 6y' = 60t^2 \quad , \quad y(0) = y'(0) = y''(0) = 0 \qquad (2)$$

If we using Sumudu transform, and the initial conditions we fid that:

$$u^2 F''(u) + 4uF'(u) + 2F(u) = 120u^2$$

Which is the same equation (1), and again Sumudu transform fails to solve this equation.

Now by applying Elzaki transform to equation (2) and making use of the initial conditions, we get:

$$T''(u) = 120u^3 \qquad and \quad T(u) = 6u^5 + c_1 u + c_2$$

By using the initial conditions we find: $c_1 = c_2 = 0$, then $T(u) = 6u^5$ and $y(t) = t^3$

This is the exact solution of equation (2).

Conclusion

In this paper, we have introduced the modified version of Sumudu and Laplace transforms, namely Elzaki transform for solving differential equations with variable coefficients which was not solved by Sumudu transform. It has been shown that Elzaki transform is a very effective method for solving initial value problems compared with Sumudu transform. In a large domain the accurate convergence of Elzaki transform will be discussed in the coming research.

Appendix

ELzaki Transform of some Functions

$f(t)$	$\mathrm{E}[f(t)] = T(u)$
1	u^2
t	u^3
t^n	$n!\,u^{n+2}$
$t^{a-1}/\Gamma(a), a > 0$	u^{a+1}
e^{at}	$\dfrac{u^2}{1-au}$
te^{at}	$\dfrac{u^3}{(1-au)^2}$
$\dfrac{t^{n-1}e^{at}}{(n-1)!}$, $n = 1, 2, \ldots.$	$\dfrac{u^{n+1}}{(1-au)^n}$
$\sin at$	$\dfrac{au^3}{1+a^2u^2}$
$\cos at$	$\dfrac{u^2}{1+a^2u^2}$
$\sinh at$	$\dfrac{au^3}{1-a^2u^2}$
$\cos\mathrm{h}\, at$	$\dfrac{au^2}{1-a^2u^2}$
$e^{at}\sin bt$	$\dfrac{bu^3}{(1-au)^2+b^2u^2}$
$e^{at}\cos bt$	$\dfrac{(1-au)u^2}{(1-au)^2+b^2u^2}$
$t\sin at$	$\dfrac{2au^4}{1+a^2u^2}$
$J_0(at)$	$\dfrac{u^2}{\sqrt{1+au^2}}$
$H(t-a)$	$u^2 e^{-\frac{a}{u}}$
$\delta(t-a)$	$u\, e^{-\frac{a}{u}}$

References

[1] Tarig M. Elzaki, The New Integral Transform “Elzaki Transform” Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 57-64.
[2] Tarig M. Elzaki & Salih M. Elzaki, Application of New Transform “Elzaki Transform” to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 65-70.
[3] Tarig M. Elzaki & Salih M. Elzaki, On the Connections Between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, ISSN 0973-4554 Volume 6, Number 1(2011),pp. 1-11.
[4] Tarig M. Elzaki & Salih M. Elzaki, On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554 Volume 6, Number 1(2011),pp. 13-18.
[5] Tarig M. Elzaki, Adem Kilicman, Hassan Eltayeb. On Existence and Uniqueness of Generalized Solutions for a Mixed-Type Differential Equation, Journal of Mathematics Research, Vol. 2, No. 4 (2010) pp. 88-92.
[6] Tarig M. Elzaki, Existence and Uniqueness of Solutions for Composite Type Equation, Journal of Science and Technology, (2009). pp. 214-219.
[7] Lokenath Debnath and D. Bhatta. Integral transform and their Application second Edition, Chapman & Hall /CRC (2006).
[8] A.Kilicman and H.E.Gadain. An application of double Laplace transform and Sumudu transform, Lobachevskii J. Math.30 (3) (2009), pp.214-223.
[9] J. Zhang, Asumudu based algorithm m for solving differential equations, Comp. Sci. J. Moldova 15(3) (2007), pp – 303-313.
[10] Hassan Eltayeb and Adem kilicman, A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, VOL, 4,2010, no.22,1089-1098
[11] Kilicman A. & H. ELtayeb. A note on Integral transform and Partial Differential Equation, Applied Mathematical Sciences, 4(3) (2010), PP.109-118.
[12] Hassan ELtayeh and Adem kilicman, on Some Applications of a new Integral Transform, Int. Journal of Math. Analysis, Vol, 4, 2010, no.3, 123-132.